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Gap Exponents for Percolation Processes with Triangle Condition

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A study is made of the gap exponents for percolation processes with the triangle condition in the subcritical region. It is show that the gaps are given by $\Delta_t = 2$ for $t = 2, 3,\ldots$. Scaling theory predicts that $P_p(|C_0| \geqslant S(p)) \simeq (p_c - p)$ and $E_p(1/|C_0|;\ |C_0| \geqslant S(p)) \simeq (p_c - p)^3$, where $S(p)$ is the typical cluster size. It is found that $(p_c - p) \lesssim P_p(|C_0 \geqslant S(p)^{1-\varepsilon}) \lesssim (p_c - p)^{1-2\varepsilon}$ and $(p_c - p)^3 \lesssim E_p(1/|C_0|; |C_0| \geqslant S(p)^{1-\varepsilon})) \lesssim (p_c - p)^{3-4\varepsilon}$.

KEY WORDS: Percolation; triangle condition; gap exponents; free energy.

1. INTRODUCTION

Let each site in $\mathbf{Z}^d$, $d \geqslant 2$, be independently occupied or unoccupied with probability p or $1-p$, respectively. We say that x is connected to y if and only if there exists a sequence of occupied sites $x_0 = x, x_1, x_2,\ldots, x_n = y$ such that each pair $x_i x_{i+1}$ is nearest neighbor. We denote $\{x \in \mathbf{Z}^d: 0 \to x\}$ the cluster of sites connected to 0. Let $X =$ the number of points x that are connected to 0 and also let $P_n(p) = P_p(X = n)$. It is our main objective in percolation theory to study the distribution of the clusters near

$$p_c = \inf\{p \in [0, 1]: P_p(X = \infty) > 0\}$$

We denote $P_\infty(p) \equiv P_p(X = \infty)$ and call it the percolation probability. It is known that $p_c \in (0, 1)$ if the dimension $d \geqslant 2$. To study this we first look at the behavior of the moments

$$E_p(X^t; X < \infty) = \sum_{n=0}^{\infty} n^t P_n(p) \qquad \text{for} \quad t = 1, 2, 3,\ldots$$

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the percolation probability

$$P_{\infty}(p) = 1 - \sum_{n=0}^{\infty} P_n(p)$$

and the free energy

$$f(p) \equiv E_p\left(\frac{1}{X}; X \geqslant 1\right) = \sum_{n=1}^{\infty} \frac{1}{n} P_n(p)$$

as the site density p approaches p_c. It was shown recently by Aizenman and Barsky[1] that $E_p(X) < \infty$ if and only if $p < p_c$. Then one can easily see that for $p < p_c$ the moments $E_p(X^t)$ are the same as the $E_p(X^t; X < \infty)$. It is clear that $E_p(X^t)$, $t = 1, 2,...$, are increasing with respect to p. In fact, they tend to ∞ if $p \uparrow p_c$ as a consequence of (1a) of the following theorem:

Theorem 1 (Aizenman and Newman[2]). Let X be the number of sites in the cluster containing 0. Then we have

$$\frac{d}{dp} E_p(X) \leqslant 2dE_p^2(X) \tag{1a}$$

$$E_p(X^t) \leqslant B_t [E_p(X)]^{2t-1} \tag{1b}$$

where $B_t = (2t-1)(2t-3) \cdots 3 \cdot 1$ and $t = 1, 2, 3,...$.

It is widely believed that the subscritical behavior of any percolation model that satisfies the triangle condition

$$(\nabla) \qquad \sum_{x,y} P_{p_c}(0 \to x)\, P_{p_c}(x \to y)\, P_{p_c}(y \to 0) < \infty$$

should be similar to the subcritical behavior of the percolation model on the Bethe lattice. The critical behavior of the percolation of the Bethe lattice is well known. In fact, the rates of decay of the moments of the finite cluster size $E_p(X^t)$, of the percolation probability $P_{\infty}(p)$, and of the "singular part" of the free energy $f_{\text{sing}}(p)$ are all known. To describe the rate of decay of a quantity $g(p)$ about p_c, we introduce the definition of the critical exponent λ as in

$$g(p) \simeq (p_c - p)^{\lambda} \qquad \text{for} \quad p < p_c$$

or

$$g(p) \simeq (p - p_c)^{\lambda} \qquad \text{for} \quad p > p_c$$

if there are positive constants $K_1, K_2 > 0$ so that

$$K_1(p_c - p)^\lambda \leqslant g(p) \leqslant K_2(p_c - p)^\lambda$$

or similarly

$$K_1(p - p_c)^\lambda \leqslant g(p) \leqslant K_2(p - p_c)^\lambda$$

We now define the critical exponents α, β, γ, and Δ_t, respectively, for the "singular part" of the free energy, the percolation probability, the mean of the finite cluster size, and the gap moments as follows:

$$f_{\text{sing}}(p) \simeq (p_c - p)^{2-\alpha}$$

$$P_\infty(p) \simeq (p - p_c)^\beta$$

$$E_p(X) \simeq (p_c - p)^{-\gamma}$$

$$E_p(X^{t+1})/E(X^t) \simeq (p_c - p)^{-\Delta_{t+1}}$$

An analysis of percolation on the Bethe lattice (e.g., Durrett[4]) reveals that $\alpha = -1$, $\beta = 1$, $\gamma = 1$, and $\Delta_{t+1} = 2$ for all $t = 1, 2, 3,\dots$. In this paper we study the subcritical vehavior of the percolation models in $\mathbf{Z}^d$ satisfying the triangle condition (∇) by looking at the critical exponents of the mentioned quantities. In Section 2 we discuss the influence of the triangle condition on the gap exponents and show $\Delta_t = 2$. In Section 3 we discuss the exponents α and β and give a proof of a proposition concerning these exponents.

2. TRIANGLE CONDITION AND GAP EXPONENTS

It can be easily seen from Theorem 1 that $\gamma \geqslant 1$, $\Delta_t \leqslant 2\gamma$ and that $\gamma = 1$, $\Delta_t = 2$ are saturated values for percolation processes. To see how $\Delta_t = 2\gamma$ can be obtained, we need to compare $E_p(X^t)$ and $E_p^{2t-1}(X)$. We observe that

$$E_p(X) = \sum_{x_1,\dots,x_t} \tau_{t+1}(x_0, x_1,\dots, x_t)$$

and

$$E_p^{2t-1}(X) = B_t^{-1} \sum_{x_1,\dots,x_t} T_{t+1}(x_0, x_1,\dots, x_t)$$

where

$$\tau_{t+1}(x_0, x_1,\dots, x_t) = P_p(x_0 \to x_1, x_2,\dots, x_t)$$

$$T_{t+1}(x_0, x_1,\dots, x_t) = \sum_G{}' \sum_{y_1,\dots,y_{t-1}} \prod_{(z,z') \in E(G)} \tau_2(z, z')$$

[in the definition of T_{t+1} the sum $\sum_G'$ is over all the connected tree graphs G with external vertices $\{x_0, x_1,..., x_t\}$, internal vertices $\{y_1,..., y_{t-1}\}$, and the set of edges $E(G)$]. Thus, to compare $E_p(X^t)$ and $E_p^{2t-1}(X)$ we may need to compare τ_{t+1} and T_{t+1}. As a matter of fact, Aizenman and Newman[2] obtained the result (1b) by first deriving their tree graph inequality that

$$\tau_{t+1}(x_0, x_1,..., x_t) \leqslant T_{t+1}(x_0, x_1,..., x_t)$$

and then summing over $x_1, x_2,..., x_t$ on both sides of the inequality. They further conjectured that in systems satisfying the (∇) condition, there is also a lower bound of the form

$$(*) \qquad \tau_{t+1}(x_0,..., x_t) \geqslant \delta^{t-1} T_{t+1}(x_0,..., x_t) \qquad \text{for some} \quad \delta > 0$$

If $(*)$ holds, then we can have the other bound of (1b):

$$E_p(X^t) \geqslant \delta^{t-1} E_p^{2t-1}(X)$$

by simply summing over $x_1,..., x_t$ on both sides of $(*)$. Therefore, the values $\Delta_t = 2\gamma$ are implied by the tree structure of the higher connectivity functions T_{t+1}, provided the vertex strengths $G_t = \tau_t / T_t$ are bounded below. Moreover, the values $\Delta_t = 2$ would be obtained if $\gamma = 1$. It is already known that the value $\gamma = 1$ can be attained for systems with the (∇) condition, as shown in the following result.

Theorem 2 (Aizenman and Newman[2]). If the triangle condition (∇) holds, then $\exists \delta > 0$ such that

$$\frac{d}{dp} E_p(X) \geqslant \delta E_p^2(X) \tag{2}$$

Indeed, from the two theorems above we can easily show that $\exists K_1, K_2 > 0$ such that

$$K_1(p_c - p)^{-1} \leqslant E_p(X) \leqslant K_2(p_c - p)^{-1} \tag{3}$$

While we do not know how to prove the conjecture $(*)$, we can still deduce the main conclusion on the gap exponents, $\Delta_t = 2$, from the tree structure of just the three-point function (which is the mechanism behind Theorem 2) or in fact just from inequality (2), by applying the following general result.

Theorem 3 (Durrett and Nguyen[5]). There exists a positive constant K_3 such that

$$S(p) = \frac{E_p(X^2)}{E_p(X)} \geqslant K_3 \frac{1}{E_p^2(X)} \left[\frac{d}{dp} E_p(X)\right]^2 \tag{4}$$

To see why by above theorem implies $\Delta_2 = 2$, we apply (2) and (3) in (4) to obtain

$$S(p) \geqslant K_3\, \delta^2 E_p^2(X) \geqslant K_3\, \delta^2 K_1^2 (p_c - p)^{-2} = K_4 (p_c - p)^{-2}$$

On the other hand, from (1b) we have

$$S(p) \leqslant B_2 [E(X)]^2 \leqslant B_2 K_2^2 (p_c - p)^{-2} = K_5 (p_c - p)^{-2}$$

This shows that $\Delta_2 = 2$. We will show in the next section that $\Delta_t = 2$ for all $t = 2, 3, 4,...$, by applying inductively (1b). The $S(p)$ defined in Theorem 3 is known as the typical cluster size and plays a very important role in the scaling theory for percolation, as we will see in the discussion about α and β in the next section.

3. DISCUSSION ON α AND β

In this section we discuss α and β and show a proposition concerning these exponents. In the course of doing this, we prove that $\Delta_t = 2$ for $t = 2, 3,...$. Note that if we think of Δ_1 as $\gamma + \beta$ and Δ_0 as $2 - \alpha - \beta$ and if we believe that the gaps are constant—i.e., $\Delta_t = \Delta$ for $t = 0, 1, 2,...$—then we would expect that $\beta = 1$ and $\alpha = -1$ for systems with the triangle condition. In general, it is known from Chayes and Chayes[9] that $\beta \leqslant 1$ and from Aizenman and Barsky[1] that $\beta(\delta - 1) \geqslant 1$, where the critical exponent δ is defined as in

$$P_{p_c}(X \geqslant n) \simeq n^{-1/\delta}$$

According to Barsky,[3] $\delta = 2$, provided the (∇) condition holds. This amounts to $\beta = 1$. At this point, as far as we know, the question of whether $\alpha = -1$ is still open. Further, we do not know whether the free energy is singular at p_c. Physicists have suggested that the singular part of the free energy should come from the tail

$$f_s(p) = \sum_{n \geqslant S(p)} n^{-1} P_n(p)$$

where $S(p)$ is the typical cluster size (see, e.g., Essam[6] or Stauffer[8]). They introduce scaling theory, which suggests

$$(**) \qquad P_n(p) \sim n^{-1/\delta} \exp[-n/S(p)] \qquad \text{for} \quad p < p_c$$

Assuming (**), we see that for $p < p_c$

$$\begin{aligned} f_s(p) &\equiv \sum_{n \geqslant S(p)} n^{-1} P_n(p) \\ &\sim \sum_{n=S(p)}^{\infty} n^{-1/\delta - 1} \exp[-n \mid S(p)] \\ &\sim S(p)^{-(1/\delta + 1)} \int_1^{\infty} x^{-1/\delta - 1} e^{-x}\, dx \\ &= \text{const} \times S(p)^{-(1/\delta + 1)} \end{aligned}$$

Also,

$$\begin{aligned} P_{\geqslant S}(p) &\equiv \sum_{n \geqslant S(p)} P_n(p) \\ &\sim \sum_{n=S(p)}^{\infty} n^{-1/\delta} \exp[-n \mid S(p)] \\ &\sim S(p)^{-1/\delta} \int_1^{\infty} x^{-1/\delta} e^{-x}\, dx \\ &= \text{const} \times S(p)^{-1/\delta} \end{aligned}$$

This, together with the (∇) condition, shows that $f_s(p) \simeq (p_c - p)^3$ and $P_{\geqslant S}(p) \simeq (p_c - p)$, since we already know that $\delta = 2$ and $S(p) \simeq (p_c - p)^{-2}$. In the following proposition we show that, even without the assumption (**), this is "almost" correct.

Proposition. Assume that the triangle condition (∇) holds. Then, given a positive integer t, there exists a neighborhood $N(p_c) \equiv (p_c(t), p_c)$ such that

$$A_t (p_c - p)^{1 - 2/t} \geqslant P_p(X \geqslant S(p)^{1 - 1/t}) \geqslant \tilde{A}_t (p_c - p) \tag{5}$$

and

$$C_t (p_c - p)^{3 - 4/t} \geqslant E_p(1/X;\, X \geqslant S(p)^{1 - 1/t}) \geqslant \tilde{C}_t (p_c - p)^3 \tag{6}$$

where A_t, $\tilde{A}_t$, C_t, and $\tilde{C}_t$ are some positive constants depending on t.

Note that as $t \uparrow \infty$, $S^{1 - 1/t}(p) \to S(p)$; hence, by dominated convergence

$$P_p(X \geqslant S(p)^{1 - 1/t}) \to P_p(X \geqslant S(p))$$
$$E_p(1/X;\, X \geqslant S(p)^{1 - 1/t}) \to E_p(1/X;\, X \geqslant S(p))$$

Since the critical exponents of the two quantities in the proposition tend to 1 and 3, we expect that the critical exponents of their limits should be the limiting values above and we think of them as the representatives of the percolation probability and the singular part of the free energy for the case of $p < p_c$.

We start the proof with the following easy observation.

Lemma 1. The following inequality holds:

$$\frac{E(X^t)}{E(X^{t-1})} \geqslant \frac{E(X^{t-1})}{E(X^{t-2})}$$

Proof. This is an easy consequence of the Cauchy–Schwartz lemma:

$$E^2(X^{t-1}) = E^2(X^{t/2} X^{t/2-1}) \leqslant E(X^t)\, E(X^{t-2})$$

Thus by induction we have

$$\frac{E_p(X^t)}{E_p(X^{t-1})} \geqslant \frac{E_p(X^{t-1})}{E_p(X^{t-2})} \geqslant \cdots \geqslant \frac{E_p(X^2)}{E_p(X)} = S(p) \tag{7}$$

Hence

$$E_p(X^t) \geqslant E_p(X)\, S(p)^{t-1} \tag{8}$$

On the other hand, from (1b) we obtain

$$\begin{aligned} \frac{E_p(X^t)}{E_p(X^{t-1})} &\leqslant \frac{B_t [E_p(X)]^{2t-1}}{E_p(X)\, S(p)^{t-2}} \\ &\leqslant \frac{B_t [K_2(p_c - p)^{-1}]^{2t-1}}{K_1(p_c - p)^{-1} [K_4(p_c - p)^{-2}]^{t-2}} \leqslant \tilde{B}_t (p_c - p)^{-2} \end{aligned} \tag{9}$$

where $\tilde{B}_t$ is some positive constant depending on t. Expressions (7) and (9) show that $\Delta_t = 2$ for all $t = 2, 3, 4, \ldots$. By a similar reasoning, we can show that there exist positive constants $\tilde{K}_t$, $\tilde{\tilde{K}}_t$ so that

$$\tilde{K}_t (p_c - p)^{-2t+1} \leqslant E_p(X^t) \leqslant \tilde{\tilde{K}}_t (p_c - p)^{-2t+1} \tag{10}$$

Now we turn our attention to the proof of (5). Half of (5) is an easy consequence of the Chebyshev inequality:

$$\begin{aligned} P_p(X \geqslant S^{1-1/t}(p)) &\leqslant E_p(X)/S^{1-1/t}(p) \leqslant K_2(p_c - p)^{-1} [K_4^{-1}(p_c - p)^2]^{1-1/t} \\ &= K_2 K_4^{-1+2/t} (p_c - p)^{1-2/t} \end{aligned}$$

To show the other half, fix an integer t. We have

$$S(p)^{t-1} \leqslant [K_5(p_c-p)^{-2}]^{t-1} = K_5^{t-1}(p_c-p)^{-2t+2} \leqslant \tfrac{1}{2}\tilde{K}_t(p_c-p)^{-2t+1}$$

if p is close enough to p_c inside the neighborhood $N(p_c) = (p_c(t), p_c)$. If so, then by (10)

$$S(p)^{t-1} \leqslant \tfrac{1}{2}E_p(X^t)$$

Hence, we have, for $p \in N(p_c)$,

$$\begin{aligned} P_p(X \geqslant S(p)^{1-1/t}) &= P_p(X^t \geqslant S(p)^{t-1}) \geqslant P_p(X^t \geqslant \tfrac{1}{2}\, E_p(X^t)) \\ &\geqslant \frac{\frac{1}{4}E_p^2(X^t)}{\frac{1}{4}E_p^2(X^t) + \mathrm{Var}(X^t)} \end{aligned}$$

(by the one-sided Markov inequality[7])

$$\geqslant \frac{1}{4}\,\frac{E_p^2(X^t)}{E_p(X^{2t})} \geqslant \frac{1}{4}\,\frac{\tilde{K}_t^2(p_c-p)^{-4t+2}}{\tilde{\tilde{K}}_{2t}^2(p_c-p)^{-4t+1}} = \tilde{A}_t(p_c-p)$$

where $\tilde{A}_t = \frac{1}{4}\tilde{K}_t^2\tilde{\tilde{K}}_{2t}^{-2}$. This completes the proof of (5). The proof of (6) can be proved easily from (5) as follows:

$$\begin{aligned} \sum_{n \geqslant S(p)^{1-1/t}} \frac{1}{n}\, P_n(p) &\leqslant \frac{1}{S(p)^{1-1/t}}\, P_p(X \geqslant S(p)^{1-1/t}) \\ &\leqslant A_t(p_c-p)^{1-2/t}[K_4^{-1}(p_c-p)^2]^{1-1/t} \\ &= C_t(p_c-p)^{3-4/t} \end{aligned}$$

where $C_t = A_t K_4^{-1+1/t}$; and by Cauchy–Schwartz

$$\begin{aligned} \sum_{n \geqslant S(p)^{1-1/t}} \frac{1}{n}\, P_n(p) &\geqslant \frac{P_p^2(X \geqslant S(p)^{1-1/t})}{\sum_{n \geqslant S(p)^{1-1/t}} P_n(p)} \geqslant \frac{P_p^2(X \geqslant S(p)^{1-1/t})}{E_p(X)} \\ &\geqslant \frac{\tilde{A}_t^2(p_c-p)^2}{K_1(p_c-p)^{-1}} \\ &= \tilde{C}_t(p_c-p)^3 \end{aligned}$$

where $\tilde{C}_t = \tilde{A}_t^2 K_1^{-1}$. Q.E.D.

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